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Rational Cherednik algebra / rational $\boxed{\text{double affine Hecke alg}}$

I) Motivation
II) Description & Examples
III) Rep. theory (f.dim reps)
IV) KZ-functor
V) Calogero-Moser space.

I) $\mathfrak{h} = \mathbb{C}^n$.

$S_n \curvearrowright X = \mathfrak{h} \times \mathfrak{h}^*$, symplectic. $\left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right]$.
finite

Rmk:
In general:
G = finite gp acting symplectically on $V \leftarrow$ symp. v.sp/$\mathbb{C}$
Ex: • G: finite Coxeter gp
• $\Gamma \subseteq sl_2(\mathbb{C}) \curvearrowright \mathbb{C}^2$
• $G = S_n \ltimes \Gamma^n \curvearrowright (\mathbb{C}^2)^n$

Study orbifold X/S_n : singular in general.
$\qquad \| $
$(\mathbb{C}^2)^n/S_n$: n unordered pts on $\mathbb{C}^2$.

Approach 1:

$\qquad$ smooth, quasi-projective.

$\mathrm{Hilb}^n(\mathbb{C}^2) = \{ \text{ideals } I \subseteq \mathbb{C}[x,y] \mid \dim(\mathbb{C}[x,y]/I) = n \}$.

$\downarrow$ resolution of singularities

X/S_n

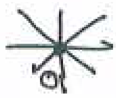
$m = \langle x, y \rangle$ max ideal of $\mathbb{C}[x,y]$

Examples: $n=2$.

$$\begin{array}{ccccccc} D & \in & \mathrm{Hilb}^2(\mathbb{C}^2) & \supseteq & \mathbb{P}(\mathbb{C}^2) = \{ \langle ax+by \rangle + m^2 \} \\ \downarrow & & \downarrow & & \downarrow \\ D & \in & (\mathbb{C}^2)^2/S_2 & \ni & 2[0] \end{array}$$

$a, b \in \mathbb{C}$

$D = \{a \neq b\}$



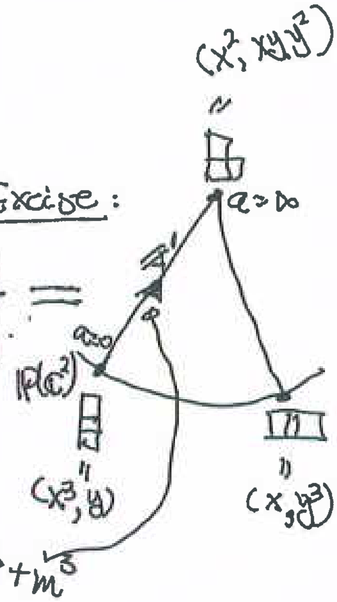
$n=3$.

$$\begin{array}{ccccccc} D & \in & \mathrm{Hilb}^3(\mathbb{C}^2) & \supseteq & ? = \{ I \mid \sqrt{I} = m, \ \dim(\mathbb{C}[x,y]/I) = 3 \} \\ \downarrow & & \downarrow & & \downarrow \\ D & \in & (\mathbb{C}^2)^3/S_2 & \ni & 3[0] \end{array}$$

$D = \{a \neq b \neq c\}$

$=$

Excise:



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Approach 2

replace $\mathbb{C}[X]^{S_n}$ by $\underbrace{\mathbb{C}[X] \rtimes S_n}$
comm.   non-commutative

Morita equivalence

Actually: $e = \frac{1}{|S_n|} \sum_g g$ idempotent

$$\mathbb{C}[X]^{S_n} = \underbrace{e\,(\mathbb{C}[X] \rtimes S_n)\,e}_{\text{spherical subalg.}}$$

RCA: $H_{t,c}$ is a universal deformation of $\mathbb{C}[X] \rtimes S_n$!
parameters are t, c.

Relation Thms

Mckay corresp.

① [Haiman, 2002] $D^b(\mathrm{Coh}\, \mathrm{Hilb}^n(\mathbb{C}^2)) \cong D^b(\mathbb{C}[X] \rtimes S_n\text{-mod})$

② [Kashiwara-Rouquier, 2007]

A_c : quantization of $\mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2)}$

certain A_c - mod $\cong H_{1,c}$ - f.g modules with conditions.

③ [Bezrukavnikov - Finkelberg - Ginzburg, 2006]

Over char p, $c \in \mathbb{F}_p$.

$D^b(\mathrm{Coh}(\underbrace{\mathrm{Hilb}^n(\mathbb{A}^2)}_{\text{over } \overline{\mathbb{F}}_p}))$ $\underset{\text{roughly.}}{\simeq}$ $D^b(H_{1,c}\text{-mod})$ over $\overline{\mathbb{F}}_p$

sheaves of coherent $\mathcal{H}_c$-mod on $\mathrm{Hilb}^n(\mathbb{A}^2)$, where $\mathcal{H}_c$: some Azumaya alg on $\mathrm{Hilb}^n(\mathbb{A}^2)$.

Motivation 2:

$$E = \boxed{\text{elliptic curve } \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}}$$

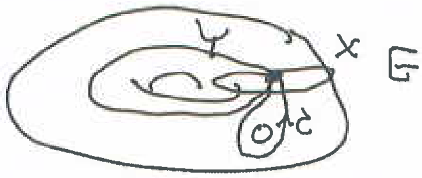
$$\mathrm{Conf}_n(E) = \{(a_1, \dots, a_n) \in E^n \mid a_i \neq a_j\} / S_n$$

$\pi_1(\mathrm{Conf}_n(E))$ = elliptic braid gp. B^{ell}

Example: $n=2$. $\overline{\mathrm{Conf}_2(E)} = (E \setminus 0)/S_2 \qquad z \mapsto -z$

(Relation: $\mathrm{Conf}_2(E) = (E \times E \setminus \Delta)/S_2 \leftarrow E$).

$\pi_1(E \setminus 0) \cong$



$\overset{gp}{\cong} \langle X, Y, C \mid XYX^{-1}Y^{-1} = C \rangle$

Introduce T (half loop)

$$B^{ell} \underset{gp}{\cong} \left\langle X, Y, T \;\middle|\; \begin{array}{l} ① \; TXT = X^{-1} \\ ② \; T^{-1}YT^{-1} = Y^{-1} \\ ③ \; Y^{-1}X^{-1}YXT^2 = 1 \end{array} \right\rangle$$

<u>Cherednik:</u> (for A_1) ~ 2000

$$\boxed{\mathcal{H}^{DAHA}(q, v)} = \mathbb{C}[B^{ell}] \big/ (T - v)(T + v^{-1}) = 0.$$

$$\overset{alg}{\cong} \langle T, X^{\pm}, Y^{\pm} \rangle \Big/ \begin{array}{l} \bullet \; ① \\ \bullet \; ② \\ \bullet \; Y^{-1}X^{-1}YXT^2 = q \\ \bullet \; (T - v)(T + v^{-1}) = 0 \end{array}$$

$X = e^{\hbar x}$

$Y = e^{\hbar y}$

$q = e^{\hbar} \qquad T = s e^{\hbar c s} \qquad \hbar \to 0$ degeneration

$v = e^{\hbar c}$

$$H_{1,c}(A_1) \overset{alg}{\cong} \langle x, y, s \rangle \Big/ \begin{array}{l} \bullet \; sx = -xs \\ \bullet \; sy = -ys \\ \bullet \; s^2 = 1 \\ \bullet \; [y, x] = 1 - 2cs \end{array}$$

$x \circlearrowleft \mathbb{Z}_2$

$y \circlearrowleft \mathbb{Z}_2$

$\mathbb{Z}_2$

[Etingof-Ginzburg ~2001] $\boxed{\text{rational DAHA}}$

II) <u>Def:</u> For $t, c \in \mathbb{C}$ [type A_{n-1}]

$$H_{t,c} := \underset{\text{free alg}}{\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle} \rtimes S_n \Big/$$

- $[x_i, x_j] = 0 \qquad [y_i, y_j] = 0$
- $[y_i, x_j] = c\, s_{ij} \qquad i \neq j$
- $[y_i, x_i] = t - \sum_{j \neq i} c\, s_{ij}$.

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Examples: ① $c=0$, $t=0$

$$H_{0,0} = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes S_n$$

② $c=0$, $t=1$

$$H_{1,0} = \mathrm{Diff}(\mathfrak{h}) \rtimes S_n$$

↖ alg of differential operators on $\mathfrak{h}$

③ $\forall \lambda \in \mathbb{C}^*$

$$H_{t,c} \underset{\text{algebra}}{\cong} H_{\lambda t, \lambda c}$$ (So usually work with $H_{1,c}$)

Let $\mathfrak{h}^{reg} = \mathbb{C}^n \setminus \{x_i = x_j\}_{i \neq j}$.

$\mathrm{Diff}(\mathfrak{h}^{reg}) \rtimes S_n$:

Special elts: (1)

$$D_i = \frac{\partial}{\partial x_i} - c \sum_{i \neq j} \frac{1}{x_i - x_j}(1 - s_{ij})$$ Dunkl operator.

Thm $\boxed{[EG]}$

① $\exists$ an embedding

$$H_{1,c} \hookrightarrow \mathrm{Diff}(\mathfrak{h}^{reg}) \rtimes S_n$$

$x_i \longmapsto x_i$

$y_i \longmapsto D_i$

$s_{ij} \longmapsto s_{ij}$. $\Rightarrow$ $[D_i, D_j] = 0$

② • $H_{1,c}^{loc} \cong \mathrm{Diff}(\mathfrak{h}^{reg}) \rtimes S_n$

$H_{1,c}$ is a mod over $\mathbb{C}[\mathfrak{h}]$

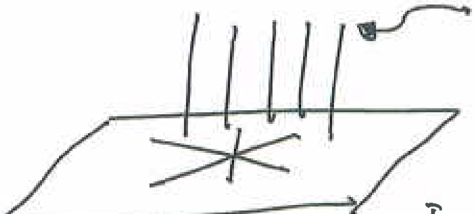
loc. w.r.t $\prod_{i,j}(x_i - x_j)$

$\mathfrak{h}^{reg} \rightsquigarrow (E^n)^{reg}$ gen.

elliptic Dunkl operator of Etingof-Ma, 2008.

• $H_{0,c}^{loc} \cong \mathbb{C}[\mathfrak{h}^{reg} \oplus \mathfrak{h}] \rtimes S_n$

Think:



$\mathfrak{h}^{reg}$

$\mathbb{C}[S_n] \times \mathfrak{h}^{reg} = E$

Conn. on the bundle:

$\nabla = d + A$: flat, S_n-equiv.

A: 1-form with coeff in $\Omega^1(\mathfrak{h}^{reg}) \otimes \mathbb{C}[S_n]$

In general: $E \to \Omega^1(\mathfrak{h}^{reg}) \otimes E$

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(III) Rep. theory of $H_{t,c}$.

Fundamental: • classify f. dim reps of $H_{t,c}$
• Compute dim / char formula of the reps.

Case 1: $t=0$

let $Z_{0,c} \subseteq H_{0,c}$ center

Thm: $Z_{0,c} \cong \underbrace{eH_{0,c}e}_{\text{spherical}} \xrightarrow{\text{deforms}} \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{S_n}$

$H_{0,c} \xrightarrow{\text{deforms}} \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\rtimes S_n$ □

$\Rightarrow$ $H_{0,c}$ is f. generated over $Z_{0,c}$

For $\forall \underset{\text{irreducible}}{M} \circlearrowleft H_{0,c}$, $Z_{0,c}$ acts by a character $\chi: Z_{0,c} \to \mathbb{C}$

$\Rightarrow$ M is f.dim.

Thm ① [Etingof - Ginzburg ~2001]

Any irrep reps of $H_{0,c}$ has dim $n!$, and is isom. to the regular rep $\mathbb{C}[S_n]$ as S_n-module;

Acturally: they're parametrized by $\mathrm{Spec}(Z_{0,c})$ ← CM space.

②. [Gordan ~2003] $Z' := \mathbb{C}[\mathfrak{h}^*]^{S_n}\otimes\mathbb{C}[\mathfrak{h}]^{S_n} \hookrightarrow Z_{0,c}$

The irred. reps of $H_{0,c}$, Z' acts by 0, are indexed by $\lambda\vdash n$.

$L(\lambda)$. From ①: $L(\lambda) \overset{S_n\text{-mod}}{\cong} \mathbb{C}[S_n]$.

③ [Griffeth ~2012, relies on Haiman's $n!$ thm ~2002]

$\mathrm{ch}(\mathrm{gr}\, L(\lambda)) = \underbrace{\tilde{H}_\lambda(q,t)}_{\text{transformed}}$ Macdonald poly.

A mod of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]\rtimes S_n$.

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<u>Rmk</u>: [Haiman, 2002]

$$\begin{array}{ccc} \mathcal{P}_{I_\mu} & \subseteq & \mathcal{P} \\ \downarrow & & \downarrow \\ I_\mu & \in & \mathrm{Hil}^n(\mathbb{C}^2) \ni D \end{array}$$

$\mathcal{P}$: Procesi bundle, rk $n!$

$\mathcal{P}_D$ = regular reps of S_n

$(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^2$

I_μ ← Partition monomial ideals

[$n!$ ~~conjecture~~ Theorem: $\mathcal{P}$ is a vector bundle.]

$\mathcal{P}_{I_\mu} \circlearrowleft S_n \times (\mathbb{C}^*)^2$.

Thm [Haiman]

$$\boxed{\text{bigraded ch}(\mathcal{P}_{I_\mu}) = \tilde{H}_\mu(z; q, t)}$$

transformed Macd. poly.

$\text{bigraded ch}(\mathcal{P}_{I_\mu}) \in \underbrace{R(S_n)}_{\text{reps ring of } S_n}(q, t)$

$\tilde{H}_\mu(z; q, t) \in$ Sym poly ring (q, t) in $z = z_1, z_2 \cdots$

$$\tilde{H}_\mu(z; q, t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t) \underbrace{s_\lambda(z)}_{\text{Schur function}}$$

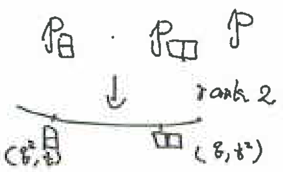
$\Rightarrow$ Macdonald positivity conj: $\tilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q^{\pm}, t^{\pm}]$

Reproved by Gordan, Bezrukavnikov - Finkelberg.

<u>Examples</u> $n=2$

$$\left.\begin{array}{l} \tilde{H}_{(1,1)} = s_{(1,1)} + q\, s_{(2)} \\ \tilde{H}_{(2)} = s_{(1,1)} + t\, s_{(2)} \end{array}\right\} \text{ character of } \mathcal{P}_{(2)} \ \& \ \mathcal{P}_{(1,1)}$$

$\mathrm{Hil}^2(\mathbb{C}^2)_0 = \mathbb{P}(\mathbb{C}^2)$



<u>Ques</u>: $\begin{array}{c} \mathcal{P} \\ \downarrow \\ \mathrm{Hil}^2(\mathbb{C}^2) \end{array} \quad \begin{array}{c} \mathcal{P}_D = \mathcal{O}_D = \mathbb{C}[x,y]/I_D \\ \downarrow \\ \ni D \end{array}$

as S_2-reps

$\text{triv} \otimes \mathbb{C} \oplus \text{sign} \otimes \mathbb{C}$.

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$\boxed{\text{Griffeth's work:}}$

$n!$ thm $\Longleftrightarrow$ $\mathrm{gr}\, L(\lambda) \cong \mathcal{P}_\lambda$, all λ.

Rmks: 2d, $\frac{\infty}{2}$ Grass $\subseteq$ $\mathbb{P}$(Fusion bundle) ← Combinatoric inf. is in Macdonald polynomial

$\downarrow$ construct!

$\mathrm{Hil}^n(\mathbb{C}^2)$

Case 2: When $t=1$ $Z_{1,c}$ is trivial!

- Unreasonable & Hard to focus on f. dim reps
- Standard approach (like Cat $\mathcal{O}$ for semisimple $\mathfrak{g}$)

  Define a cat $\mathcal{O}$ of $H_{1,c}$. $\mathcal{O} \supseteq$ f.d rep

Complication $\begin{cases} ① & \text{f. dim reps are not completely reducible} \\ ② & \text{classifications \& char formulas for f. dim reps are known in special cases} \end{cases}$

(S_n known!
other complex reflection gps ?)

About Cat $\mathcal{O}$:
[GGOR, 2003]

$M \in \mathrm{Cat}(\mathcal{O})$.

- M is f. generated
- $\mathfrak{h}$ acts locally nilpotently.

Examples of modules in $\mathcal{O}$

① f. dim reps

② $H_{1,c} \underset{\text{vector space}}{=\!=} \mathbb{C}(\mathfrak{h}) \otimes S_n \otimes \mathbb{C}(\mathfrak{h}^*)$

Take any $S_n \curvearrowright E$ f. dim.

$\mathbb{C}(\mathfrak{h}) \otimes S_n \curvearrowright E$ s.t $\mathfrak{h}$ acts by 0.

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(8)

Verma modules $\Delta(E) := \underbrace{H_{1,t}}_{\text{inf. dim}} \otimes_{\mathbb{C}(\mathfrak{h})\otimes S_n} E$ (as vector space $\mathbb{C}[\mathfrak{h}^*]\otimes E$)

Std thm: $\Delta(E)$ has a unique irreducible quotient $L(E)$.

$\{L(E) \mid E \in \mathrm{irr}(S_n)\}$ is a complete list of irreducible objects in $\mathcal{O}$.

Q: When is $L(E)$ f. dim?

Thm [Berest – $\boxed{\text{Etingof}}$ – Ginzburg, 2002]

(1) $H_{1,c}(S_n)$ has a non-trivial f. dim reps

Elliptic affine Springer fibre

$\Rightarrow$ $\boxed{c = \pm \frac{r}{n}, \ \gcd(r, n) = 1.}$

(2) If $c > 0$, the unique f. d irreducible reps of $H_{1,c}$ is $L(\mathrm{triv})$

If $c < 0$, the unique f. d irreducible reps of $H_{1,c}$ is $L(\mathrm{sign})$

Rmks: $H_c \cong H_{-c}$ $\rightsquigarrow$ $\mathcal{O}_c \simeq \mathcal{O}_{-c}$

$\sigma \longleftrightarrow \mathrm{sign}(\sigma)\sigma$

$x \longleftrightarrow x$

$y \longleftrightarrow y$

$L(\mathrm{triv}) \longleftrightarrow L(\mathrm{sign})$

(3) A char. formula is given.

In particular, $\dim L = r^{n-1}$ dep. on c.

Rmks:

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1) Generalize $\rightsquigarrow$ W: finite gp acting on $V \leftarrow$ symplectic vector space

$S_n \rightsquigarrow$ Ex: $S_n \ltimes (\mathbb{Z}_\ell)^n \curvearrowright \mathfrak{h} \oplus \mathfrak{h}^*$

ℓ-th roots of unity in $\mathbb{C}^*$

$H_{1,c} \rightsquigarrow$ Cyclotomic rat'l DAHA

[Rouquier-Shan-Varagnolo-Vasserot, 2013]

2) Proof uses KZ-functor!

Powerful tool to study Cat $\mathcal{O}$ of $H_{1,c}$.

3) Geometric approach

- Varagnolo-Vasserot [2009]. $H^{DAHA} \curvearrowright K(\widehat{Sp})$ — affine Springer fibers
  $H^{\text{trig DAHA}} \curvearrowright H^*(\widehat{Sp})$
- Oblomkov-Yun [~2016]. $H_{1,c} \curvearrowright grH^*(\widehat{Sp})$
  $c = \frac{r}{n}$, $(r,n)=1$.

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{gives all irreducible f. dim modules}}$

(IV) KZ-functor: The BLACK BOX theorem

$$KZ: \mathcal{O}_c \longrightarrow \underbrace{\mathcal{H}_c}_{\text{Hecke alg of } S_n}(S_n)\text{-mod}$$

$$M \longmapsto M_x$$

- exact
- fully faithful on projectives
- $\rightsquigarrow$ induces an equivalence $\mathcal{O}_c / \mathcal{O}_c^{tor} \cong \mathcal{H}_c(S_n)\text{-mod}$,

How to define it?

$\mathcal{O}_c^{tor} = \{M \in \mathcal{O}_c \mid \mathbb{C}[\mathfrak{h}^*] \text{ acts torsion}\}$

$M \in \mathcal{O}_c$, up to localization $M \circlearrowleft D(\mathfrak{h}^{reg}) \rtimes S_n$.

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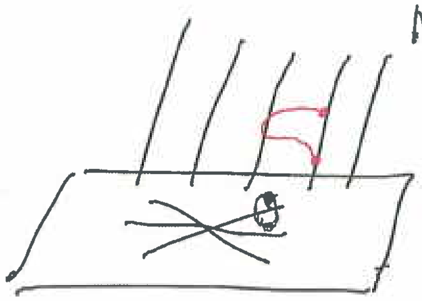

$\mathfrak{h}^{reg}$

$M_x \times \mathfrak{h}^{reg}$,

$\nabla = d + A$, flat.

A: 1-form, coeff are $\mathrm{End}(M_x) \otimes \mathcal{O}(\mathfrak{h}^{reg})$.

$$\begin{tikzcd} \mathbb{C}[\pi_1(\mathfrak{h}^{reg}/S_n)] \arrow[r] \arrow[d] & \mathrm{End}(M_x) \\ \mathcal{H}_c(S_n) \arrow[ur, "\exists"] & \end{tikzcd}$$

$\rightsquigarrow$ monodromy.

OR: think:

$$\begin{tikzcd} \mathcal{O}_c \arrow[r, "\mathrm{Res}"] \arrow[dr, "KZ"'] & \mathrm{Loc}(\mathfrak{h}^{reg}/S_n) \arrow[d, "\mathrm{mon.}"] \\ & \mathrm{Rep}(\mathcal{H}_c(S_n)) \end{tikzcd}$$

Thm [GGOR] $c < 0$.

let $\Delta(\lambda)$ be the Verma mod ass. to the Specht module for S_n.

Then:

- $KZ(\Delta(\lambda)) = S^{(\lambda)} \leftarrow$ Specht mod for $\mathcal{H}(S_n)$
- $KZ(\underbrace{L}_{\text{simple}}(\lambda)) = S^{(\lambda)}/\mathrm{Rad} \rightsquigarrow$ irreducible quotient of $S^{(\lambda)}$.

Recall:

- Specht module of S_n:

  all irreducible reps labeled by $\lambda \vdash n$.

- Specht module of $\mathcal{H}(S_n)$:

  a deformation of that of S_n.

V) $\boxed{\text{Calogero - Moser space.}}$

$\underline{t=0}$.

$\underline{\text{Recall}}$: $GL_n(\mathbb{C}) \curvearrowright T^* Mat_n(\mathbb{C})$ $\circlearrowright$ n

$$\mu : T^* Mat_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})^*$$
$$(X, Y) \mapsto [X,Y] \cup\!\!| \Lambda$$

$\mu^{-1}(\Lambda)$ $\quad$ $\Lambda := \{A \in \mathfrak{sl}_n(\mathbb{C}) \mid rk(A+I) = 1\}$

$$= \underbrace{GL_n \begin{pmatrix} n-1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}}_{GL_n\text{-orbit}}$$

[Kazhdan, Kostant, Sternberg] 1978

Def: Calogero Moser space.

$$C_n = \mu^{-1}(\Lambda) /\!/ G \qquad \text{[the action is free]}$$

Smooth, symplectic variety dim $2n$, connected.

$$\underset{\substack{\text{dense}\\ \text{open}}}{C_n \supseteq} \mathcal{U} \cong \overset{\text{symplectic var}}{T^*(\mathfrak{h}^{reg}/S_n)}$$

$$\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \begin{pmatrix} y_1 & & \frac{1}{x_i - x_j} \\ & \ddots & \\ & & y_n \end{pmatrix} \right) \xleftarrow{\cong} (x_1, \dots x_n, y_1, \dots y_n)$$

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[$\boxed{\text{CM integrable system}}$ skip?]

$\{H_1, \cdots, H_n\}$ functions on $T^*\mathrm{Mat}(\mathbb{C})$, s.t. $\{H_i, H_j\} = 0$.
alg. indep

$\rightsquigarrow$ decent to C_n, if they are still alg indep.

since $n = \frac{\dim C_n}{2}$ $\Rightarrow$ get an integrable system.

In our example:

$$T^*\mathrm{Mat}_n(\mathbb{C}) = \{(X, Y) \in \mathrm{Mat}(\mathbb{C})^2\}, \quad \omega = \mathrm{Tr}(dY \wedge dX)$$

Let $\{H_i = \mathrm{Tr}(Y^i)\}$ $\rightsquigarrow$ gives integrable system on C_n

Example: On opens,

$$H_1 = \mathrm{Tr}(Y) = \sum_{i=1}^n y_i$$

$$H_2 = \mathrm{Tr}(Y^2) = \sum_i y_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$$] skip!

<u>Thm</u>: let $Z_{0,c} \subseteq H_{0,c}$. (center)

$\boxed{\mathrm{Spec}(Z_{0,c}) \cong C_n.}$ as a symplectic variety.

Thm [Nakajima 1999]

$\mathrm{Hil}_n(\mathbb{C}^2)$ is C^∞-diffeomorphic to C_n

<u>Proof of Thm</u>:

Match on opens:

$$\mathrm{Spec}(Z_{0,c}) = \{\chi : Z_{0c} \to \mathbb{C}\}$$

$=$ moduli of irreducible reps of $H_{0,c}$

Recall:

$$H_{0,c} \hookrightarrow \mathbb{C}[x_1, \ldots x_n; y_1, \cdots, y_n, \tfrac{1}{x_i - x_j}] \rtimes S_n$$

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$\mathrm{Spec}(Z_{\mathcal{H}_0}) \supseteq \mathcal{U} = \{ E \mid x_i - x_j \text{ acts invertibly} \}$

$= \{ E = E_{(\lambda,\mu)} \mid (\lambda,\mu) \in \mathfrak{h}^{reg} \times \mathfrak{h}^* \quad E_{(\lambda,\mu)} = \text{space of functions on the } S_n\text{-orbit } \mathcal{O}_{(\lambda,\mu)} \}$

$\mathcal{U} \xleftrightarrow{\cong} V$

$E_{(\lambda,\mu)} \longmapsto \left(\begin{smallmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{smallmatrix} \right), \left(\begin{smallmatrix} \mu_1 & & \frac{1}{\lambda_i - \lambda_j} \\ & \ddots & \\ \cdots & & \mu_n \end{smallmatrix} \right)$